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Asymptotics of the Christoffel Functions on a Simplex in $\mathbb{R}^d$

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We obtain the asymptotics of the Christoffel functions for certain Jacobi-type weight functions on the standard simplex in $\mathbb{R}^d$. We also establish a sharp upper bound of the asymptotics for a large class of weight functions on the simplex. © 1999 Academic Press

Key Words: orthogonal polynomials in several variables on simplex; Christoffel function; asymptotics.

1. INTRODUCTION

The purpose of this paper is to study the asymptotics of the Christoffel functions with respect to a weight function on the standard simplex Σ^d of $\mathbb{R}^d$,

$$\Sigma^d = \{\mathbf{x} \in \mathbb{R}^d : x_1 \geqslant 0, ..., x_d \geqslant 0, 1 - x_1 - \cdots - x_d \geqslant 0\}.$$

Let Π^d be the space of polynomials in d variables and let Π_n^d be the subspace of polynomials of degree at most n. For a weight function W defined on a region Ω in $\mathbb{R}^d$, the nth Christoffel function, $\Lambda_n(W)$, can be defined by

$$\Lambda_n(W; \mathbf{x}) = \min_{P(\mathbf{x}) = 1,\, P \in \Pi_n^d} \int_\Omega P^2(\mathbf{y})\, W(\mathbf{y})\, d\mathbf{y}. \tag{1.1}$$

It has a close relation with orthogonal polynomials in several variables. Let us denote by $\mathscr{P}_n^d$ the space of homogenous polynomials of degree n on $\mathbb{R}^d$ and let $r_n^d = \dim \mathscr{P}_n^d$; it follows that $r_n^d = \binom{n+d-1}{n}$. Let $\{P_k^n\}$, $1 \leqslant k \leqslant r_n^d$, and $0 \leqslant n < \infty$, denote one family of orthonormal polynomials with respect to W that forms a basis of Π_n^d, where the superscript n means that $P_k^n \in \Pi_n^d$.

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The reproducing kernel, $\mathbf{K}_n(W)$, of the space Π_n^d with respect to W is defined by

$$\mathbf{K}_n(W; \mathbf{x}, \mathbf{y}) = \sum_{k=0}^{n} \mathbf{P}_k(W; \mathbf{x}, \mathbf{y}) \qquad \text{and}$$

$$\mathbf{P}_k(W; \mathbf{x}, \mathbf{y}) = \sum_{j=1}^{r_k^d} P_j^k(\mathbf{x}) \, P_j^k(\mathbf{y}). \tag{1.2}$$

The function $\mathbf{P}_n(W)$ is the reproducing kernel of the subspace spanned by P_k^n, $1 \leqslant k \leqslant r_n^d$. An alternative definition of the Christoffel function $\Lambda_n(W)$ is

$$\Lambda_n(W; \mathbf{x}) = 1/\mathbf{K}_n(W; \mathbf{x}, \mathbf{x}). \tag{1.3}$$

We note that the definition of $\mathbf{P}_n(W)$ or $\mathbf{K}_n(W)$ is independent of the particular choice of the orthonormal basis (see, for example, [8, p. 250]).

Because of their important role in the theory of orthogonal polynomials in one variable, the Christoffel functions have been studied intensively; in particular, their asymptotics have been established for large classes of measures on $\mathbb{R}$; see, for example, the survey [4]. On the contrary, relative little is known for the Christoffel functions in several variables; we refer to [6, 8] for the applications of $\Lambda_n(W)$ in the theory of orthogonal polynomials in several variables and to [2, 6, 7] for the few cases where the asymptotics of $\Lambda_n(W)$ have been established. The results obtained so far, however, suggest that the asymptotics of $\Lambda_n(W)$ take the form

$$\lim_{n \to \infty} \binom{n+d}{d} \Lambda_n(W; \mathbf{x}) = W(\mathbf{x})/W_0(\mathbf{x}), \tag{1.4}$$

where W_0 is an analogy of the Chebyshev weight function associated with the domain under consideration. Indeed, the above equation has been established for the Chebyshev weight function $W(\mathbf{x}) = \prod (1 - x_i^2)^{-1/2}/\pi^d$ on $[-1, 1]^d$ [6] and the weight functions $W(\mathbf{x}) = w_\alpha(1 - |\mathbf{x}|^2)^{\alpha - 1/2}$ on the unit ball $B^d = \{\mathbf{x} \colon |\mathbf{x}|_2 \leqslant 1\}$, for which $\alpha = 0$ corresponds to the Chebyshev weight on B^d [2]. Moreover, for a large class of radial weight functions on B^d, the inequality

$$\limsup_{n \to \infty} \binom{n+d}{d} \Lambda_n(W; \mathbf{x}) \leqslant W(\mathbf{x})/W_0(\mathbf{x}) \tag{1.5}$$

has been proved in [7], making use of a compact formula of $K_n(W)$ for the Chebyshev weight function in [9].

In contrary to the one dimension, the geometry of the support set of weight functions can be very intricate in higher dimensional spaces. At this point, there seems to be little understanding on how this complexity reflects on the theory of orthogonal polynomials in several variables, in particular, on the asymptotics of the Christoffel functions. To better assess the situation, we need to study a few examples in detail. In the present paper we study the asymptotics of the Christoffel functions for weight functions defined on the simplex. We shall prove the asymptotics relation (1.4) for the Chebyshev weight function $(x_1 \cdots x_d(1-x_1- \cdots -x_d))^{1/2}$ and a few other weight functions on Σ^d, and prove the inequality (1.5) for a large class of weight functions on Σ^d. The proof will be based on a compact formula of the reproducing kernel for the Chebyshev weight function, which was discovered only recently in [10].

2. THE CHRISTOFFEL FUNCTION FOR THE CHEBYSHEV WEIGHT FUNCTION

We start with the compact formula of $\mathbf{P}_n$ derived in [10] for the weight function

$$W_\alpha(\mathbf{x}) = w_\alpha x_1^{\alpha_1 - 1/2} \cdots x_d^{\alpha_d - 1/2}(1-|\mathbf{x}|_1)^{\alpha_{d+1}-1/2}, \qquad \alpha_i \geqslant 0, \tag{2.1}$$

on the simplex Σ^d, where w_α is a normalization constant such that $\int_{\Sigma^d} W_\alpha \, d\mathbf{x} = 1$. Let $|\mathbf{x}|_1 = |x_1| + \cdots + |x_d|$ for $\mathbf{x} \in \mathbb{R}^d$. The compact formula of $\mathbf{P}_n(W_\alpha)$ takes the form

$$\mathbf{P}_n(W_\alpha; \mathbf{x}, \mathbf{y}) = \frac{2n + |\alpha|_1 + (d-1)/2}{2^{d+1}\,(|\alpha|_1 + (d-1)/2)} \times \int_{[-1,1]^{d+1}} C_{2n}^{(|\alpha|_1 + (d-1)/2)}(\sqrt{x_1 y_1}\, t_1 + \cdots + \sqrt{x_{d+1} y_{d+1}}\, t_{d+1}) \times \prod_{i=1}^{d+1} c_{\alpha_i}(1-t_i^2)^{\alpha_i - 1}\, d\mathbf{t},$$

where $\mathbf{x}, \mathbf{y} \in \Sigma^d$, $x_{d+1} = 1 - |\mathbf{x}|_1$ and $y_{d+1} = 1 - |\mathbf{y}|_1$, and the constant c_μ is defined by $c_\mu = 1/\int_{-1}^{1} (1-t^2)^{\mu-1}\, dt$. Moreover, if one of the $\alpha_i \to 0$, then the formula holds upon using the limit

$$\lim_{\mu \to 0} c_\mu \int_{-1}^{1} f(t)(1-t^2)^{\alpha-1}\, dt = [f(1) + f(-1)]/2.$$

In particular, when $\alpha = 0$, we end up with the formula

$$\mathbf{P}_n(W_0; \mathbf{x}, \mathbf{y}) = \frac{2n + (d-1)/2}{2^d (d-1)} \times \sum_{\varepsilon \in \{-1,1\}^{d+1}} C_{2n}^{((d-1)/2)}\,(\sqrt{x_1 y_1}\, \varepsilon_1 + \cdots + \sqrt{x_{d+1} y_{d+1}}\, \varepsilon_{d+1}),$$

where $\mathbf{x}, \mathbf{y} \in \Sigma^d$, $x_{d+1} = 1 - |\mathbf{x}|_1$ and $y_{d+1} = 1 - |\mathbf{y}|_1$. We call the weight function

$$W_0(\mathbf{x}) = w_0 x_1^{-1/2} \cdots x_d^{-1/2} (1 - |\mathbf{x}|_1)^{-1/2}, \qquad \text{where} \quad w_0 = \pi^{(d+1)/2} / \Gamma((d+1)/2)$$

the Chebyshev weight function on Σ^d. The compact formula has been used in [10] to prove that the expansion of a continuous function in the Fourier orthogonal series with respect to W_α is uniformly Cesáro (C, δ) summable if and only if $\delta > |\alpha|_1 + (d-1)/2$, when one $\alpha_i = 0$. The asymptotics of $\mathbf{P}_n(W_0)$ will also be derived from the compact formula. We need the asymptotic formula for $C_n^{(\lambda)}$ from [5, Theorem 8.21.8, p. 196], which we state as

Lemma 2.1. *For $\lambda > 0$, $x = \cos\theta$,*

$$C_n^{(\lambda)}(x) = \frac{\Gamma(\lambda + 1/2)\, \Gamma(n + 2\lambda)\, 2^\lambda}{\Gamma(2\lambda)\, \Gamma(n + \lambda + 1/2)\, \Gamma(1/2)} \times \left[\frac{1}{(\sin\theta)^\lambda} n^{-1/2} \cos\left((n+\lambda)\,\theta - \frac{\lambda\pi}{2} \right) + \mathcal{O}(n^{-3/2}) \right] \tag{2.2}$$

for $0 < \theta < \pi$; in particular,

$$C_n^{(\lambda)}(x) = \mathcal{O}(n^{\lambda - 1}), \qquad 0 < \theta < \pi, \tag{2.3}$$

where the bound for the error term holds uniformly in $[\varepsilon, \pi - \varepsilon]$.

Before we state our result, we need one more definition. Recall that the simplex Σ^d is defined by $d+1$ inequalities: $x_1 \geqslant 0, \ldots, x_d \geqslant 0$ and $1 - |\mathbf{x}|_1 \geqslant 0$ for $\mathbf{x} \in \mathbb{R}^d$. A *k-dimensional face* of Σ^d, $0 \leqslant k \leqslant d$, contains elements of Σ^d for which exactly $d-k$ inequalities become equalities. In particular, if $k = d$, then none of the inequalities becomes equality, so that the (unique) d-dimensional face of Σ^d is the interior of Σ^d. We also note that a 0-dimensional face is one of the vertices of the simplex Σ^d. We shall denote the union of k-dimensional faces of Σ^d by Σ_k^d for $0 \leqslant k \leqslant d$, and we also use $\operatorname{Int} \Sigma^d$ for Σ_d^d.

Theorem 2.2. *If $d \geqslant 2$, then for all $\mathbf{x} \in \Sigma_k^d$*

$$\lim_{n \to \infty} \frac{1}{\binom{n+d-1}{n}} \mathbf{P}_n(W_0; \mathbf{x}, \mathbf{x}) = 2^{d-k}, \qquad 0 \leqslant k \leqslant d.$$

Proof. Setting $\mathbf{y} = \mathbf{x}$ in the compact formula of $\mathbf{P}_n(W_0)$ and using the fact that $x_{d+1} = 1 - x_1 - \cdots - x_d$ and that $C_{2n}^{(\lambda)}$ is an even function, we have

$$\mathbf{P}_n(W_0; \mathbf{x}, \mathbf{x}) = \frac{2n+(d-1)/2}{2^d(d-1)}$$

$$\times \sum_{\varepsilon \in \{-1,1\}^{d+1}} C_{2n}^{((d-1)/2)}(x_1\varepsilon_1 + \cdots + x_{d+1}\varepsilon_{d+1})$$

$$= \frac{2n+(d-1)/2}{2^{d-1}(d-1)} \left[C_{2n}^{((d-1)/2)}(1) \right.$$

$$\left. + \sum_{k=1}^{d} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant d} C_{2n}^{((d-1)/2)}(1 - 2x_{i_1} - \cdots - 2x_{i_d}) \right]$$

If $\mathbf{x} \in \Sigma_k^d$, $k > 0$, then it has exactly $d-k$ components zero, say $x_{k+1} = \cdots = x_d = 0$, and other k components positive and satisfying, say, $1 - x_1 - \cdots - x_k > 0$; in particular, it follows that $1 < 1 - 2x_1 - \cdots - 2x_k < 1$. Therefore, for $\mathbf{x} \in \Sigma_k^d$, there are exactly

$$\binom{d-k}{1} + \binom{d-k}{2} + \cdots + \binom{d-k}{d-k} = 2^{d-k} - 1$$

many terms in the sum over $1 \leqslant i_1 < \cdots < i_k \leqslant d$ equal to $C_{2n}^{((d-1)/2)}(1)$ and the rest of the terms are in the order of $\mathcal{O}(n^{(d-3)/2})$ as $n \to \infty$ by (2.3) in Lemma 2.1. Thus, for $\mathbf{x} \in \Sigma_k^d$, we obtain

$$\mathbf{P}_n(W_0; \mathbf{x}, \mathbf{x}) = \frac{2n+(d-1)/2}{2^{d-1}(d-1)} [2^{d-k} C_{2n}^{((d-1)/2)}(1) + \mathcal{O}(n^{(d-3)/2})].$$

Using the fact that [5, (4.7.3), p. 80]

$$C_n^{(\lambda)}(1) = \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)\,\Gamma(2\lambda)} \qquad \text{and} \qquad \frac{\Gamma(n+\lambda+1)}{\Gamma(n+1)} = n^\lambda(1 + \mathcal{O}(n^{-1})),$$

we conclude that

$$\lim_{n\to\infty} \frac{1}{\binom{n+d-1}{n}} \mathbf{P}_n(W_0; \mathbf{x}, \mathbf{x})$$

$$= \lim_{n\to\infty} \frac{2n+(d-1)/2}{2^{d-1}(d-1)} \cdot 2^{d-k} \cdot \frac{\Gamma(d)(2n)^{d-2}\,(1+\mathcal{O}(n^{-1}))}{\Gamma(d-1)\,n^{d-1}(1+\mathcal{O}(n^{-1}))}$$

$$= 2^{d-k},$$

whenever $d \geqslant 3$, where we have used the fact that $\Gamma(\lambda+1)=\lambda\Gamma(\lambda)$. If $k=0$, then $\mathbf{x} \in \Sigma_0^d$ is one of the vertices of Σ^d. The formula of $\mathbf{P}_n(W_0)$ shows that

$$\mathbf{P}_n(W_0, \mathbf{x}, \mathbf{x}) = \frac{2n+(d-1)/2}{2^d (d_1)} 2^{d+1} C_{2n}^{((d-1)/2)}(1),$$

from which the desired result follows easily as in the case of $k>0$. ∎

For $d=1$, the function $\mathbf{P}_n(W_0, \mathbf{x}, \mathbf{x})$ reduces to a multiple of $T_n^2(x) = \cos n\theta$, $x=\cos\theta$; hence, the above limit does not hold for $d=1$. Although we have a compact formula of $\mathbf{P}_n(W_\alpha; \mathbf{x}, \mathbf{x})$ for all $\alpha \in \mathbb{R}_+^d$, it is not clear how to use it to obtain the asymptotic as that in Theorem 2.2.

THEOREM 2.3. *If $d \geqslant 2$, then for all $\mathbf{x} \in \Sigma_k^d$*

$$\lim_{n\to\infty} \frac{1}{\binom{n+d}{n}} \mathbf{K}_n(W_0; \mathbf{x}, \mathbf{x}) = 2^{d-k}, \qquad 0 \leqslant k \leqslant d.$$

Proof. We use the fact that if $b_n \to \infty$ as $n \to \infty$, then (cf. [3, p. 414])

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}}.$$

It then follows from (1.2) that

$$\lim_{n\to\infty} \frac{\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{x})}{\binom{n+d}{n}} = \lim_{n\to\infty} \frac{\mathbf{P}_n(W_0; \mathbf{x}, \mathbf{x})}{\binom{n+d}{n} - \binom{n-1+d}{n-1}} = \lim_{n\to\infty} \frac{\mathbf{P}_n(W_0; \mathbf{x}, \mathbf{x})}{\binom{n+d-1}{n}}.$$

Hence, the desired result follows from Theorem 2.2. ∎

COROLLARY 2.4. *If $d \geqslant 2$, then for all $\mathbf{x} \in \Sigma_k^d$*

$$\lim_{n\to\infty} \binom{n+d}{n} \Lambda_n(W_0; \mathbf{x}) = 2^{k-d}, \qquad 0 \leqslant k \leqslant d.$$

In particular, the limit is 1 for $\mathbf{x} \in \operatorname{Int} \Sigma^d$. We note that the limit also holds for $d=1$. For $\alpha \in 2\mathbb{N}^d$, the asymptotics of $\Lambda_n(W_\alpha)$ will follow from a general result in the following section.

3. ASYMPTOTICS OF THE CHRISTOFFEL FUNCTIONS

As in the case of the unit ball B^d, the knowledge of $\Lambda(W_0)$ on Σ^d allows us to derive a general result on the asymptotics of the Christoffel functions.

In this section we prove an inequality (1.5) for a large class of weight functions on Σ^d. The proof is based on an approximation identity, which is of interest in itself. We define a linear operator, $L_n(f)$, on the class of continuous functions on Σ^d as

$$L_n(f; \mathbf{x}) = \frac{1}{\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{x})} \int_{\Sigma^d} [\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{y})]^2 f(\mathbf{y}) W_0(\mathbf{y}) \, d\mathbf{y}.$$

Clearly this is a positive linear operator. Moreover, it preserves constant functions by the orthogonality and the definition of $\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{y})$; in particular, $L_n(1) = 1$, which shows that L_n is an approximate identity. We have

THEOREM 3.1. *Let f be a bounded function on Σ^d, such that f is continuous in the interior of Σ^d. Then*

$$\lim_{n \to \infty} \frac{1}{\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{x})} \int_{\Sigma^d} [\mathbf{K}_n(\mathbf{x}, \mathbf{y})]^2 f(\mathbf{y}) W_0(\mathbf{y}) \, d\mathbf{y} = f(\mathbf{x}), \qquad \mathbf{x} \in \operatorname{Int} \Sigma^d.$$

Moreover, the convergence holds uniformly on each compact set contained in Int Σ^d.

Proof. From the identity [5, p. 83, (4.7.29)],

$$(2k + \lambda) \, C_{2k}^{(\lambda)}(x) = \lambda(C_{2k}^{(\lambda+1)}(x) - C_{2k-2}^{(\lambda+1)}(x)),$$

where $C_{-2}^{(\lambda+1)} = 0$, it follows that

$$\sum_{k=0}^{n} \frac{2k + \lambda}{\lambda} C_{2k}^{(\lambda)}(x) = C_{2n}^{(\lambda+1)}(x).$$

Together with the compact formula of $\mathbf{P}_n(W_0)$ and the definition of $\mathbf{K}_n(W_0)$ in (1.2), we have the compact formula,

$$\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{x})$$
$$= \frac{1}{2^{d-1}} \sum_{\varepsilon \in \{-1, 1\}^{d+1}} C_{2n}^{((d+1)/2)} (\sqrt{x_1 y_1} \varepsilon_1 + \cdots + \sqrt{x_{d+1} y_{d+1}} \varepsilon_{d+1}).$$

From the fact that $L_n(1; \mathbf{x}) = 1$, it follows readily that

$$I_n(\mathbf{x}) := \left| \frac{1}{\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{x})} \int_{\Sigma^d} [\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{y})]^2 f(\mathbf{y}) W_0(\mathbf{y}) \, d\mathbf{y} - f(\mathbf{x}) \right|$$
$$\leqslant \frac{1}{\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{x})} \int_{\Sigma^d} [\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{y})]^2 \, |f(\mathbf{y}) - f(\mathbf{x})| \, W_0(\mathbf{y}) \, d\mathbf{y}.$$

Let $\|f\|_\infty$ denote the uniform norm of f on Σ^d and let $\omega_{\mathbf{x}}(f)$ be the local modulus of continuity of f, at a point $\mathbf{x} \in \operatorname{Int} \Sigma^d$, defined by

$$\omega_{\mathbf{x}}(f, \varepsilon) = \sup_{|\mathbf{x}-\mathbf{y}|_1 \leqslant \varepsilon} \{|f(\mathbf{x}) - f(\mathbf{y})| : \mathbf{y} \in \operatorname{Int} \Sigma^d\}.$$

Evidently, if f is continuous on $\operatorname{Int} \Sigma^d$, then $\omega_{\mathbf{x}}(f; \varepsilon) \to 0$ as $\varepsilon \to 0$. For $\mathbf{x} \in \operatorname{Int} \Sigma^d$, we choose $\varepsilon > 0$ such that the set $\Sigma_\varepsilon(\mathbf{x}) := \{\mathbf{y} \in \Sigma^d \colon |\mathbf{y} - \mathbf{x}|_1 < \varepsilon\}$ is inside Σ^d. We then have

$$\begin{aligned} I_n(\mathbf{x}) \leqslant{} & \omega_{\mathbf{x}}(f; \varepsilon) \frac{1}{\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{x})} \int_{\Sigma_\varepsilon(\mathbf{x})} [\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{y})]^2 \, W_0(\mathbf{y}) \, d\mathbf{y} \\ & + \frac{1}{\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{x})} \int_{\Sigma^d \setminus \Sigma_\varepsilon(\mathbf{x})} [\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{y})]^2 \, |f(\mathbf{x}) - f(\mathbf{y})| \, W_0(\mathbf{y}) \, d\mathbf{y} \\ \leqslant{} & \omega_{\mathbf{x}}(f; \varepsilon) + 2 \, \|f\|_\infty \frac{1}{\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{x})} \\ & \times \int_{\Sigma^d \setminus \Sigma_\varepsilon(\mathbf{x})} [\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{y})]^2 \, W_0(\mathbf{y}) \, d\mathbf{y}. \end{aligned}$$

Hence, by the compact formula of $\mathbf{K}(W_0; \mathbf{x}, \mathbf{x})$ and Theorem 2.3, it suffices to prove that

$$\begin{aligned} \lim_{n \to \infty} & \frac{1}{\binom{n+d}{n}} \int_{\Sigma^d \setminus \Sigma_\varepsilon(x)} [C_{2n}^{((d+1)/2)}(\sqrt{x_1 y_1} \, \varepsilon_1 + \cdots + \sqrt{x_{d+1} y_{d+1}} \, \varepsilon_{d+1})]^2 \\ & \times W_0(\mathbf{y}) \, d\mathbf{y} = 0, \end{aligned}$$

where $\varepsilon_i = \pm 1$. Let

$$t(\mathbf{x}, \mathbf{y}) = \varepsilon_1 \sqrt{x_1 y_1} + \cdots + \varepsilon_d \sqrt{x_d y_d} + \varepsilon_{d+1} \sqrt{x_{d+1} y_{d+1}}.$$

Since $|x_i - y_i| = |(\sqrt{x_i} - \sqrt{y_i})(\sqrt{x_i} + \sqrt{y_i})| \leqslant 2 \, |\sqrt{x_i} - \sqrt{y_i}|$ for $\mathbf{x}, \mathbf{y} \in \Sigma^d$, we have

$$\begin{aligned} |t(\mathbf{x}, \mathbf{y})| & \leqslant \sqrt{x_1 y_1} + \cdots + \sqrt{x_d y_d} + \sqrt{1 - |\mathbf{x}|_1} \sqrt{1 - |\mathbf{y}|_1} \\ & = 1 - \frac{1}{2} \left[(\sqrt{1 - |\mathbf{x}|_1} - \sqrt{1 - |\mathbf{y}|_1})^2 + \sum_{k=1}^{d} (\sqrt{x_i} - \sqrt{y_i})^2 \right] \\ & \leqslant 1 - \frac{1}{2} \sum_{k=1}^{d} (\sqrt{x_i} - \sqrt{y_i})^2 \end{aligned}$$

$$\leqslant 1-\frac{1}{8}\sum_{k=1}^{d}|x_i-y_i|^2$$

$$\leqslant 1-\frac{1}{8d}|\mathbf{x}-\mathbf{y}|_1^2,$$

where the last inequality follows from the Cauchy inequality, $|\mathbf{x}-\mathbf{y}|_1 \leqslant \sqrt{d}\,|\mathbf{x}-\mathbf{y}|_2$. If $\mathbf{y}\in\Sigma^d\backslash\Sigma_\varepsilon(\mathbf{x})$, then $|\mathbf{x}-\mathbf{y}|_1\geqslant\varepsilon$; hence, we have $|t(\mathbf{x},\mathbf{y})|\geqslant 1-(1/2d)\,\varepsilon^2$. In particular, this shows that

$$1-t(\mathbf{x},\mathbf{y})^2\geqslant 1-\left(1-\frac{1}{8d}\varepsilon^2\right)^2 := \rho^2>0, \qquad \mathbf{y}\in\Sigma^d\backslash\Sigma_\varepsilon(\mathbf{x}),$$

where we assume that ε has been chosen small enough so that $1-\varepsilon^2/8d<1$. Now, from Lemma 2.1, (2.4) and the identity [1, p. 256, (6.1.18)] of the Gamma function, it follows that

$$\frac{1}{\binom{n+d}{n}}[C_n^{(d+1/2)}(t)]^2=\frac{\Gamma(1/2)\,\Gamma((d/2)/2)}{\Gamma((d+1)/2)}\frac{2}{\pi}\frac{1}{n+(d+1)/2}$$

$$\times\left[\frac{1}{(1-t^2)^{d+1/2}}\cos^2(N\theta+\gamma)+\mathcal{O}(n^{-1})\right],$$

where $N=n+(d+1)/2$, $\gamma=(d+1)\,\pi/4$, $t=\cos\theta$. Therefore, using the estimate of $t(\mathbf{x},\mathbf{y})$ for $\mathbf{y}\in\Sigma^d\backslash\Sigma_\varepsilon(\mathbf{x})$, it follows from Lemma 2.1 that

$$\frac{1}{\binom{n+d}{n}}\int_{\Sigma^d\backslash\Sigma_\varepsilon(\mathbf{x})}[C_n^{((d+1)/2)}(\sqrt{x_1y_1}\,\varepsilon_1+\cdots+\sqrt{x_{d+1}y_{d+1}}\,\varepsilon_{d+1})]^2$$

$$\times W_0(\mathbf{y})\,d\mathbf{y}$$

$$=\mathcal{O}(1)\,\frac{1}{n\rho^{d+1}}+\mathcal{O}(n^{-2}), \qquad \mathbf{y}\in\Sigma^d\backslash\Sigma_\varepsilon(\mathbf{x}),$$

which converges to zero as $n\to\infty$. From the proof it is also clear that the convergence is uniform over a compact set inside $\operatorname{Int}\Sigma^d$. The proof is complete. ∎

We are now ready to prove the upper bound (1.5) for the asymptotics of $\Lambda_n(W)$.

THEOREM 3.2. *Let W be a nonnegative weight function on Σ^d, such that W is continuous in* $\operatorname{Int}\Sigma^d$ *and W/W_0 is bounded on Σ^d. Then*

$$\limsup_{n\to\infty}\binom{n+d}{n}\Lambda_n(W;\mathbf{x})\leqslant\frac{W(\mathbf{x})}{W_0(\mathbf{x})}, \qquad \mathbf{x}\in\operatorname{Int}\Sigma^d.$$

Proof. Because of the definition of Λ_n in (1.1) we have

$$\binom{n+d}{n} \Lambda_n(W; \mathbf{x})$$

$$\leqslant \binom{n+d}{n} \frac{1}{[\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{x})]^2} \int_{\Sigma^d} [\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{y})]^2 \, W(\mathbf{y}) \, d\mathbf{y}.$$

By Theorem 2.3, the desired result is the consequence of the limit

$$\lim_{n\to\infty} \frac{1}{\mathbf{K}_n(W_0; \mathbf{x}, \mathbf{x})} \int_{\Sigma^d} [\mathbf{K}_n(W; \mathbf{x}, \mathbf{y})]^2 \, W(\mathbf{y}) \, d\mathbf{y} = \frac{W(\mathbf{x})}{W_0(\mathbf{x})}, \quad \mathbf{x} \in \operatorname{Int} \Sigma^d,$$

which is a corollary of Theorem 3.1 with $f = W/W_0$ by the assumption on W. ∎

One immediate question is does the much stronger result of limit (1.4) holds under the assumption of the above theorem. For $W = W_0$, the limit indeed holds according to Theorem 2.3. Using an approach of Freud (cf. [6, Theorem 4.3.1]), we may prove a lower bound for lim inf $\Lambda_n(W)$ in the form of $cW(x)/W_0(x)$, where $c < 1$, for a more restricted class of weight functions. Since we do expect that the limit (1.4) will hold in a general setting, here we are content to give the following result.

THEOREM 3.3. *If $W = W_0 q^2$, where q is a polynomial of d variables, then*

$$\lim_{n\to\infty} \binom{n+d}{n} \Lambda_n(W; \mathbf{x}) = \frac{W(\mathbf{x})}{W_0(\mathbf{x})}, \qquad \mathbf{x} \in \operatorname{Int} \Sigma^d.$$

Proof. Let m denote the degree of q. By the definition of Λ_n in (1.1), we have

$$\begin{aligned}
\Lambda_n(W; \mathbf{x}) &= \min_{P(\mathbf{x})=1,\, P \in \Pi_n^d} \int_\Omega P^2(\mathbf{y}) \, W(\mathbf{y}) \, d\mathbf{y} \\
&= \min_{P(\mathbf{x})=1,\, P \in \Pi_n^d} \int_\Omega |P(\mathbf{y}) \, q(\mathbf{y})|^2 \, W_0(\mathbf{y}) \, d\mathbf{y} \\
&= q^2(\mathbf{x}) \min_{P(\mathbf{x})=1,\, P \in \Pi_n^d} \int_\Omega |P(\mathbf{y}) \, q(\mathbf{y})/q(\mathbf{x})|^2 \, W_0(\mathbf{y}) \, d\mathbf{y} \\
&\geqslant q^2(\mathbf{x}) \min_{Q(\mathbf{x})=1,\, Q \in \Pi_n^d} \int_\Omega Q^2(\mathbf{y}) \, W_0(\mathbf{y}) \, d\mathbf{y} \\
&= \frac{W(\mathbf{x})}{W_0(\mathbf{x})} \Lambda_{n+m}(W_0; \mathbf{x}).
\end{aligned}$$

Hence, using the fact that $\binom{n+m+d}{d}/\binom{n+d}{d} \to 1$ as $n \to \infty$, it follows from Corollary 2.4 that

$$\liminf_{n\to\infty} \binom{n+d}{n} \Lambda_n(W;\mathbf{x}) \geqslant \frac{W(\mathbf{x})}{W_0(\mathbf{x})}, \qquad \mathbf{x} \in \operatorname{Int} \Sigma^d.$$

The desired result follows from the above inequality and Theorem 3.2. ∎

In particular, for W_α in (2.1), we have that W_α/W_0 is the square of a polynomial if all α_i are even integers. Hence, we conclude,

Corollary 3.4. *If $\alpha \in 2\mathbb{N}^d$, that is, $\alpha_i \in 2\mathbb{N}$ for all i, then*

$$\lim_{n\to\infty} \binom{n+d}{n} \Lambda_n(W_\alpha;\mathbf{x}) = \frac{W_\alpha(\mathbf{x})}{W_0(\mathbf{x})}, \qquad \mathbf{x} \in \operatorname{Int} \Sigma^d.$$

Although the compact formula for $\Lambda_n(W_\alpha)$ is known for all α, we do not see how to prove the asymptotic formula for general α at this point. It is worth mentioning that an analog result of Theorem 3.2 has been proved for radial weight functions on the unit ball B^d in [7], and the limit (1.5) has been established for $U_\alpha(\mathbf{x}) = u_\alpha(1-|\mathbf{x}|^2)^{\alpha-1/2}$ on B^d, where u_α is a normalization constant, and U_0 serves as the corresponding Chebyshev weight function [2, 7]. The proof of these facts in [7] relied on a compact formulae for $\mathbf{P}_n(U_\alpha)$ derived in [9] for the purpose of studying summability of Fourier orthogonal series. Following the approach in [7], we may be able to prove the limit (1.5) in the case that exactly one component of α is not zero.

We conclude this note with the following remark. The results that we obtained above and in [7] indicate a similarity between the structure on Σ^d and that on B^d. This similarity is by no means accidental. In fact, there is a one-to-one correspondence between the class of orthogonal polynomials on Σ^d and the class of $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ invariant orthogonal polynomials on B^d.

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REFERENCES

1. M. Abramowitz and I. Stegun, "Handbook of Mathematical Functions," 9th ed., Dover, New York, 1970.
2. L. Bos, Asymptotics for the Christoffel function for Jacobi like weights on a ball in $\mathbb{R}^m$, *New Zealand J. Math.* **23** (1994), 99–109.

3. T. J. Bromwich, "Introduction to the Theory of Infinite Series," 2nd ed., MacMillan, London, 1926.
4. P. Nevai, Géza Freud, orthogonal polynomials and Christoffel functions: A case study, *J. Approx. Theory* **48** (1986), 3–167.
5. H. Szegö, "Orthogonal Polynomials," 4th ed., Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, RI, 1975.
6. Y. Xu, Christoffel functions and Fourier Series for multivariate orthogonal polynomials, *J. Approx. Theory* **82** (1995), 205–239.
7. Y. Xu, Asymptotics for orthogonal polynomials and Christoffel functions on a ball, *Methods Appl. Anal.* **3** (1996), 257–272.
8. Y. Xu, On orthogonal polynomials in several variables, *in* "Special Functions, q-Series, and Related Topics," Vol. 14, pp. 247–270, Fields Institute Communications, Amer. Math. Soc., Providence, 1997.
9. Y. Xu, Summability of Fourier orthogonal series for Jacobi weight on a ball in $\mathbb{R}^d$, *Trans. Amer. Math. Soc.* **351** (1999), 2439–2458.
10. Y. Xu, Summability of Fourier orthogonal series for Jacobi weights on the simplex in $\mathbb{R}^d$, *Proc. Amer. Math. Soc.* **126** (1998), 3027–3036.